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Abstract

A basket option is an option whose underlying is a portfolio of individual stock prices. Due to the unknown dependence structure between stocks, basket option pricing relies in general on approximations or numerical methods like Monte Carlo simulation. We propose a methodology for pricing basket options in a multivariate Variance Gamma model. The stock prices composing the basket are then modeled by time changed geometric Brownian motions with a common Gamma subordinator. Using the additivity property of comonotonic stop-loss premiums together with Gauss-Laguerre polynomials, we derive a closed-form expression for the basket option price as a linear combination of Black & Scholes prices. This technique manages to approximate the real basket option price in an accurate way. Furthermore, our new basket option pricing formula enables us to calibrate the multivariate VG model in a fast way provided option quotes on the components and the basket itself are available. As an illustration, we show that the multivariate VG model can closely match the observed Dow Jones index options.

1 Introduction

Nowadays, different methods to price single name options, also called vanilla options, are available and for a variety of models, the price of a vanilla call option can be calculated in a fast and efficient way. The industry standard for modeling stock price behavior is proposed in Black & Scholes (1973), where the asset price process is described by a geometric Brownian motion. This approach results in closed-form expressions for vanilla option prices, which, to a large extent, explains the popularity of this model. It is well-known that assuming a geometric Brownian motion is a too simplistic approach, in that it cannot account for the skewness and the kurtosis observed for asset log returns. Furthermore,

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market crashes like the one on Black Monday, or the default of Lehman Brothers are observed more frequently than the Black & Scholes model prescribes.

The above-mentioned shortcomings of the Black & Scholes model have led to the search for and the development of more realistic stochastic models to describe the behavior of stock prices. A flexible class of stochastic models is the class of the Lévy models; see for example Sato (1999) and Schoutens (2003) for an overview of Lévy processes and their applicability in finance. These advanced stock price models allow for jumps, excess kurtosis and skewness and, as a result, are more suitable for modeling stock price behavior than the Black & Scholes model. In Madan & Seneta (1990), the Variance Gamma process was introduced for modeling the behavior of a single stock. In Carr & Madan (1999), it was shown that in this model setting, vanilla option prices can be approximated using the FFT in an accurate and computational efficient way. Furthermore, this approach can be extended to other Lévy processes, as long as the characteristic function of the Lévy process is given in closed form.

The increased volume of multi-asset derivatives has shifted research to the question how to model the joint dynamics of a number of dependent stock prices. The most straightforward choice is to extend the univariate Black & Scholes model to the multivariate case, by using correlated Brownian motions; see e.g. Björk (1998) and Dhaene et al. (2013). This multivariate stock price model uses a lognormal distribution for the single stock prices and a Gaussian dependence structure. However, a Gaussian dependence structure is not realistic; for example, it does not allow for tail dependence. In Luciano & Schoutens (2006), a multivariate version of the Variance Gamma process was introduced. The individual stock prices are still modeled using a Variance Gamma process, but they are dependent through a common time change. This new multivariate model can benefit from the strengths of the univariate Variance Gamma process and also introduces a non-Gaussian dependence structure. Extensions of this multivariate Lévy process can be found in Moosbrucker (2006), Semeraro (2008), Leoni & Schoutens (2008), Luciano & Semeraro (2010) and Guillaume (2013).

Throughout this paper, we assume that a basket always consists of a number of stocks. The distribution of the basket is in general unknown and, as a result, basket option pricing relies on approximations or numerical methods like Monte Carlo simulation. In this paper, we search for an approximate basket option pricing formula when the stock prices composing the basket are modeled by a multivariate Variance Gamma model. Conditional on the time change, the basket is a weighted sum of correlated geometric Brownian motions. An extensive bibliography is dedicated to the problem of finding accurate approximations for basket options when dealing with correlated Brownian motions. Here, we use convex upper and lower bounds to derive approximate basket option prices; see e.g. Kaas et al. (2000) and Vanduffel et al. (2005). Having an accurate approximation for the conditional basket option price, the (unconditional) basket option price is determined in closed form using a Gauss-Laguerre quadrature formula. It turns out that the approximate basket option price is given by a weighted sum of Black & Scholes prices, where the weights and values of the Gamma subordinator are based on Gauss-Laguerre polynomials. Note also that our methodology for pricing basket options can be extended to the situation where the joint stock price dynamics can be described by time changed Brownian motions with common time change. Simulation results show that our methodology is able to closely approximate the basket option price. Furthermore, we can derive the approximate distribution function of the basket.

Using prices of traded single-asset and multi-asset derivatives, we propose a methodology for calibrating the multivariate VG model; see also Linders & Schoutens (2014*a*,*b*) Our approximate basket option pricing formula is an essential tool in order to have an efficient and fast calibration of the multivariate Variance Gamma model. Moreover, this calibration is fully market implied, i.e. only traded derivatives are used. We assume here that options on the components and on the basket are traded. This situation occurs when we consider a traded stock market index like the Dow Jones, EUROSTOXX50, S&P 500, ... A basket option is then often called an index option. The calibration is carried out in two steps. Firstly, the marginal parameters and the distribution of the common time change can be calibrated to market quotes of the vanilla options by using the Carr-Madan formula. Secondly, the availability of traded index options together with an approximate basket option pricing formula enables us to calibrate the remaining correlation parameter, where we make the simplifying assumption that all pairwise correlations are the same.

We illustrate this calibration procedure using the Dow Jones index, a price weighted stock market index consisting of 30 American stocks. Using the available vanilla and index option prices, we are able to calibrate the multivariate VG model. To test if the calibrated model is suitable for multi-asset derivative pricing, we investigate if the model index option prices are comparable to their quoted counterparts. Remarkably, we can conclude that the quoted DJ option prices can be closely matched by the calibrated multivariate VG model.

To the best of our knowledge, only a few papers have investigated the performance of different multivariate stock price models for reproducing observed vanilla and index option curves; see e.g. Jourdain & Sbai (2012) and Cont & Deguest (2013). Multivariate stock price models based on correlated VG processes and their calibration are discussed in Ballotta et al. (2014), Ballotta & Bonfiglioli (2014) and Linders & Schoutens (2014*a*). However, these models will, in general, fail to provide an accurate fit and produce an *implied correlation smile* when calibrated to index option data; see also Garcia et al. (2009), Tavin (2013) and Linders & Schoutens (2014*b*).

2 Convex order and comonotonicity

In this section, we summarize some definitions and results concerning convex order, inverse distributions and comonotonicity needed afterwards.

Given two random variables X and Y, X is said to precede Y in *convex order sense*, notation $X \preceq_{cx} Y$, if

$$\begin{cases} \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right] \\ \mathbb{E}\left[(K-X)_+\right] \leq \mathbb{E}\left[(K-Y)_+\right] \end{cases}, \qquad \text{for all } K \in \mathbb{R}. \tag{1}$$

Consider the random vector $(X_1, X_2, \ldots, X_n)$ and the weighted sum $S = \sum_{i=1}^{n} w_i X_i$, where $w_i > 0$. Assume that the marginal stop-loss premiums $\mathbb{E}\left[(X_i - K)_+\right]$ can be determined for any K. Even if we have full information about the marginal distributions, calculating the stop-loss premium $\mathbb{E}\left[(S - K)_+\right]$ is, in general, not straightforward as it requires information about the dependence among the marginals. Specifying this dependence structure can be done by choosing an appropriate copula. In most situations, the distribution of S is unknown or will be too cumbersome to work with. However, in case the components of a random vector possess a perfect positive dependence structure, the stop-loss premium of the sum S can be calculated in closed form.

The random vector $(X_1, \ldots, X_n)$ is said to be *comonotonic* if

$$(X_1, \ldots, X_n) \stackrel{\mathrm{d}}{=} \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{2}$$

where U is in the sequel a uniform $(0,1)$ r.v. and '$\stackrel{\mathrm{d}}{=}$' denotes 'equality in distribution'. If S is a weighted sum of comonotonic random variables, the stop-loss premium $\mathbb{E}\left[(S - K)_+\right]$ can be decomposed in stop-loss premiums of the marginals with appropriately chosen retentions. We state this result in Theorem 1. For a proof of this theorem, we refer to Kaas et al. (2000).

Theorem 1 (Decomposition formula) *Consider a comonotonic random vector $(X_1, X_2, \ldots, X_n)$ and denote the weighted sum by S. Assume that F_S is continuous and strictly increasing on $[0, +\infty)$. For $K \geq 0$, the stop-loss premium $\mathbb{E}\left[(S - K)_+\right]$ can be decomposed into a linear combination of stop-loss premiums of the marginals involved:*

$$\mathbb{E}\left[(S - K)_+\right] = \sum_{i=1}^{n} w_i \mathbb{E}\left[(X_i - K_i)_+\right], \tag{3}$$

where

$$K_i = F_{X_i}^{-1}\left(F_S(K)\right), \qquad \text{for } i = 1, \ldots, n, \tag{4}$$

and $F_S(K)$ satisfies the following relation:

$$\sum_{i=1}^{n} w_i K_i = K. \tag{5}$$

In case the cdf F_S is not continuous and strictly increasing, a similar decomposition formula (3) can be proven for the stop-loss premium $\mathbb{E}\left[(S - K)_+\right]$ of a comonotonic sum S, but then the expression for the strike price K_i will be slightly different. A sufficient condition for F_S to be strictly increasing and continuous is that the marginal cdf's F_{X_i} are strictly increasing and continuous. Furthermore, for appropriately chosen strikes K_i, the decomposition formula (3) remains valid for $K < 0$; see e.g. Dhaene et al. (2002a) and Chen et al. (2014).

For an extensive overview of the theory of comonotonicity, including proofs of the results mentioned in this subsection, we refer to Dhaene et al. (2002a). Financial and actuarial applications of the concept of comonotonicity are described in Dhaene et al. (2002b). An updated overview of applications of comonotonicity can be found in Deelstra et al. (2011).

3 The multivariate Variance Gamma model

Consider a finite time horizon of T years and suppose that we are currently at time 0. We introduce the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$, where $\mathbb{P}$ is the 'real-world' probability measure. The filtered probability space is often assumed to satisfy the "usual conditions" of completeness and right-continuity. Interest rates, stock and bond prices, as well as prices of other financial assets are described by stochastic processes (which are assumed to be semi-martingales) on this filtered measurable space. For simplicity in notation and terminology, we assume deterministic interest rates.

We assume a financial market where n different (dividend or non-dividend paying) stocks, labeled from 1 to n, are traded. For each stock i, its random price at time t, $0 \leq t \leq T$, is denoted by $X_i(t)$. We denote the stochastic price process of stock i by $\{X_i(t) \mid 0 \leq t \leq T\}$. Hereafter, we will always silently assume that $X_i(t) \geq 0$ and all expectations are finite.

3.1 The stock price processes

Consider the process $\{G(t) \mid t \geq 0\}$ defined on the filtered probability space, which is non-decreasing, starts at zero and has stationary and independent increments. We call $\{G(t) \mid t \geq 0\}$ a subordinator. The parameters of the process $\{G(t) \mid t \geq 0\}$ are chosen such that $\mathbb{E}[G(t)] = t$. The characteristic function of the process G is given by $\phi_G(u) = \mathbb{E}\left[\mathrm{e}^{iuG(1)}\right]$. Throughout this paper, we assume that $\{G(t) \mid t \geq 0\}$ is a Gamma process with parameters at and b, i.e.

$$f_{G(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} \mathrm{e}^{-xb}, \quad x \geq 0. \tag{6}$$

The condition $\mathbb{E}[G(t)] = t$ implies that $a = b$ and by using the notation $\nu = \frac{1}{a}$, we also have that $\mathrm{Var}[G(t)] = t\nu$. Note that our methodology remains valid for other choices of the subordinator.

In the sequel, $\{\underline{B}(t) \mid t \geq 0\}$ denotes a standard n-dimensional Brownian motion process independent from $\{G(t) \mid t \geq 0\}$. Furthermore, for $i \neq j$, we write that

$$\rho_{i,j} = \mathrm{Corr}\left[\sigma_i B_i(t), \sigma_j B_j(t+s)\right], \tag{7}$$

where

$$\rho_{i,j} \geq 0. \tag{8}$$

We consider the multivariate Variance Gamma (VG) model introduced in Luciano & Schoutens (2006). The joint risk-neutral dynamics of the stock prices are modeled as follows,

$$X_i(t) \overset{\mathrm{d}}{=} X_i(0) \exp\left\{(r - q_i + \omega_i)t + \mu_i G(t) + \sigma_i B_i(G(t))\right\}, \quad \text{for } i = 1, 2, \ldots, n, \tag{9}$$

where

$$\omega_i = \frac{1}{\nu} \log\left(1 - \frac{1}{2}\sigma_i^2 \nu - \mu_i \nu\right)$$

is the mean correction to ensure that the corresponding process $\left\{e^{-(r-q_i)t}X_i(t) \mid t \geq 0\right\}$ is a martingale. Furthermore, $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ for each i. Stock i may pay dividends and we assume that these dividends are paid continuously at a rate q_i. In case the joint dynamics of the stocks are described by (9), each stock can be written as the exponential of a Variance Gamma process.

The market basket is composed of a linear combination of the n underlying stocks. Denoting the price of the basket at time t by $S(t)$, $0 \leq t \leq T$, we have that

$$S(t) = w_1 X_1(t) + w_2 X_2(t) + \ldots + w_n X_n(t), \tag{10}$$

where w_i, $i = 1, 2, \ldots, n$, are positive weights that are fixed up front. The price of a basket call option with maturity T and strike price K is denoted by $C[K,T]$. The marginals of the basket S are modeled by a Variance Gamma process and according to Carr & Madan (1999), the pricing of vanilla options can be done in a fast and accurate way. In this paper, we describe a methodology to find an accurate approximation, denoted by $\overline{C}[K,T]$, for the real basket option price.

The notations $F_{X_i(T)}$ and $F_{S(T)}$ will be used for the time-0 cumulative distribution functions (cdf's) of $X_i(T)$ and $S(T)$. In order to avoid unnecessary overloading of the notations, we will hereafter omit the fixed time index T when no confusion is possible.

3.2 Basket options in the multivariate VG model

Consider the time-T price vector $(X_1, X_2, \ldots, X_n)$. By equation (9), we can write the distribution of the price level S of the basket at time T as follows

$$S \stackrel{\mathrm{d}}{=} \sum_{i=1}^{n} w_i X_i(0) \exp\left\{(r - q_i + \omega_i)T + \mu_i G + \sigma_i \sqrt{G} B_i(1)\right\}, \tag{11}$$

where $B_i(1) \stackrel{\mathrm{d}}{=} N(0,1)$. Conditional on G, the components in S are all lognormally distributed:

$$\ln \frac{X_i}{X_i(0)} \stackrel{\mathrm{d}}{=} N\left((r - q_i + \omega_i)T + \mu_i G, \sigma_i^2 G\right). \tag{12}$$

From (11) and (12), it follows that the conditional random variable $S \mid G$ is a weighted sum of n dependent lognormal random variables with pairwise correlations $\rho_{i,j}$.

Using the tower property, we can write the basket option price $C[K]$ as follows:

$$C[K] = \int_0^{+\infty} e^{-rT} \mathbb{E}\left[(S-K)_+ \mid G = x\right] f_G(x) \, dx. \tag{13}$$

Note that for $x \geq 0$, the quantity $e^{-rT}\mathbb{E}\left[(S-K)_+ \mid G = x\right]$ can be considered as the price of a basket call option where the underlying basket $S \mid G = x$ is a weighted sum of correlated lognormal random variables. We introduce the following notation:

$$S_x \equiv S \mid G = x, \quad \text{for } x \geq 0. \tag{14}$$

In order to find an approximation for the basket option price $C[K]$, we first approximate the integrand $\mathrm{e}^{-rT}\mathbb{E}\left[(S-K)_+ \mid G=x\right]$ using the theory of comonotonicity and afterwards the integral $\int_0^{+\infty}\mathrm{e}^{-rT}\mathbb{E}\left[(S-K)_+ \mid G=x\right] f_G(x)\mathrm{d}x$ using Gauss-Laguerre polynomials.

For the conditional basket call option price, we introduce the following notation

$$C[K;x] = \mathrm{e}^{-rT}\mathbb{E}\left[(S_x - K)_+\right].$$

Expression (13) then shows that the price $C[K]$ for a basket call option is a mixture of the synthetic basket call option prices $C[K;x]$. An extensive bibliography is dedicated to the problem of finding accurate approximations for the basket option price $C[K;x]$. An analysis of the different approximations for the pricing of basket options in the multivariate Black & Scholes model can be found in Krekel et al. (2007). In this paper, we use convex upper and lower bounds to find an approximation for $C[K;x]$. This methodology was proposed in Vyncke et al. (2004) for Asian options and in Linders (2013) for basket options.

4 The conditional basket option price

We start the search for an approximate basket option price by conditioning on the common time change G. From Section 3.2, the basket S_x can be written as a weighted sum of correlated lognormal random variables. In this section, we consider convex upper and lower bounds for this conditional sum, which result in an upper and lower bound for the basket option price as well. Under the multivariate Variance Gamma model, an explicit expression for these bounds is obtained.

4.1 A convex upper bound for the conditional sum

Throughout this section, we fix $x \geq 0$ and derive an upper bound for $C[K;x]$ by replacing the real conditional sum S_x by the random sum S_x^c, which is defined as follows

$$S_x^c = w_1 F^{-1}_{X_1|G=x}(U) + \ldots + w_n F^{-1}_{X_n|G=x}(U). \tag{15}$$

In Kaas et al. (2000), it is proven that the comonotonic sum S_x^c is a convex upper bound for the sum S_x,

$$S_x \preceq_{cx} S_x^c. \tag{16}$$

We obtain from the definition in (1) that $C[K;x]$ can be bounded from above as follows:

$$C[K;x] \leq \mathrm{e}^{-rT}\mathbb{E}\left[(S_x^c - K)_+\right]. \tag{17}$$

Combining expressions (15) and (12), we find that S_x^c is a weighted sum of comonotonic lognormal random variables. In this special case, we can determine S_x^c explicitly in terms of the marginal parameters and the cdf Φ of a standard normal distribution.

Theorem 2 *Consider a market where the assets follow the multivariate VG model (9). Then the conditional random variable S_x^c is given by the following expression:*

$$S_x^c \overset{\mathrm{d}}{=} \sum_{i=1}^{n} w_i X_i(0)\, \mathrm{e}^{(r-q_i+\omega_i)T+\mu_i x+\sigma_i\sqrt{x}\Phi^{-1}(U)}. \tag{18}$$

Its variance is given by

$$\mathrm{Var}[S_x^c] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j X_i(0)\, X_j(0)\, \mathrm{e}^{2rT+(\omega_i-q_i+\omega_j-q_j)T+(\mu_i+\mu_j)x+\frac{\sigma_i^2+\sigma_j^2}{2}x}\left(\mathrm{e}^{\sigma_i\sigma_j x}-1\right). \tag{19}$$

Proof. The marginal risk-neutral distributions are given by (12), from which we find that

$$X_i \overset{\mathrm{d}}{=} X_i(0)\exp\left\{(r-q_i+\omega_i)\,T+\mu_i G+\sigma_i\sqrt{G}\Phi^{-1}(U)\right\}.$$

If we combine this expression with Theorem 1 in Dhaene et al. (2002a), the inverse cdf $F^{-1}_{X_i|G=x}$ is given by

$$F^{-1}_{X_i|G=x}(p) = X_i(0)\exp\left\{(r-q_i+\omega_i)\,T+\mu_i x+\sigma_i\sqrt{x}\Phi^{-1}(p)\right\}. \tag{20}$$

Combining this observation with formula (15) proves (18).

The variance $\mathrm{Var}[S_x^c]$ can be written as

$$\mathrm{Var}[S_x^c] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}\left[F^{-1}_{X_i|G=x}(U), F^{-1}_{X_j|G=x}(U)\right].$$

We have that

$$\begin{aligned}\mathrm{Var}[S_x^c] = &\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j X_i(0)\, X_j(0)\, \mathrm{e}^{2rT+(\omega_i-q_i+\omega_j-q_j)T+(\mu_i+\mu_j)x} \\ &\times \mathrm{Cov}\left[\mathrm{e}^{\sigma_i\sqrt{x}\Phi^{-1}(U)}, \mathrm{e}^{\sigma_j\sqrt{x}\Phi^{-1}(U)}\right].\end{aligned} \tag{21}$$

Note that the r.v. $\mathrm{e}^{\sigma_i\sqrt{x}\Phi^{-1}(U)}$ has a lognormal distribution and $\mathbb{E}\left[\mathrm{e}^{\sigma_i\sqrt{x}\Phi^{-1}(U)}\right] = \mathrm{e}^{\frac{\sigma_i^2 x}{2}}$. The covariance in (21) can now be written as

$$\mathrm{Cov}\left[\mathrm{e}^{\sigma_i\sqrt{x}\Phi^{-1}(U)}, \mathrm{e}^{\sigma_j\sqrt{x}\Phi^{-1}(U)}\right] = \mathrm{e}^{\frac{\left(\sigma_i^2+\sigma_j^2\right)x}{2}}\left(\mathrm{e}^{\sigma_i\sigma_j x}-1\right). \tag{22}$$

Plugging (22) in expression (21) for the variance $\mathrm{Var}[S_x^c]$ proves (19). ■

We adopt the following notation:

$$V_i = V_i(0)\,\mathrm{e}^{(r-q_i+\omega_i)T+\mu_i x+\sigma_i\sqrt{x}\Phi^{-1}(U)}, \text{ for } i = 1, 2, \ldots, n, \tag{23}$$

where $x > 0$ and $V_i(0) = X_i(0)$. The vector $(V_1, V_2, \ldots, V_n)$ is a comonotonic vector where each random variable V_i has a lognormal distribution:

$$\ln\frac{V_i}{V_i(0)} \overset{\mathrm{d}}{=} N\left((r-q_i+\omega_i)\,T+\mu_i x, \sigma_i^2 x\right). \tag{24}$$

The following theorem expresses the (conditional) upper bound $\mathrm{e}^{-rT}\mathbb{E}\left[(S_x^c-K)_+\right]$ for $C[K;x]$ in terms of the marginal prices $\mathrm{e}^{-rT}\mathbb{E}\left[(V_i-K_i)_+\right]$.

Theorem 3 *Consider a market where the assets follow a multivariate VG process (9). For $x \geq 0$, the quantity* $\mathrm{e}^{-rT}\mathbb{E}\left[\left(S_x^c - K\right)_+\right]$ *is given by*

$$\mathrm{e}^{-rT}\mathbb{E}\left[\left(S_x^c - K\right)_+\right] = \sum_{i=1}^{n} w_i \mathrm{e}^{-rT}\mathbb{E}\left[\left(V_i - K_i\right)_+\right], \tag{25}$$

where K_i is defined by

$$K_i = X_i(0)\,\mathrm{e}^{(r-q_i+\omega_i)T+\mu_i x+\sigma_i\sqrt{x}\Phi^{-1}\left(F_{S_x^c}(K)\right)} \tag{26}$$

and $F_{S_x^c}(K)$ is determined such that the following relation holds:

$$\sum_{i=1}^{n} w_i K_i = K. \tag{27}$$

Proof. From the definition of V_i in (23), we find that the sum S_x^c can be written as

$$S_x^c \overset{\mathrm{d}}{=} \sum_{i=1}^{n} w_i V_i.$$

Furthermore, because the marginal cdfs F_{V_i} are strictly increasing and continuous, the cdf $F_{S_x^c}$ is also strictly increasing and continuous. By applying Theorem 1, we obtain that

$$\mathbb{E}\left[\left(S_x^c - K\right)_+\right] = \sum_{i=1}^{n} w_i \mathbb{E}\left[\left(V_i - K_i\right)_+\right],$$

where K_i follows from expressions (4) and (5). Expression (23) for V_i results in the following expression for $F_{V_i}^{-1}(p)$

$$F_{V_i}^{-1}(p) = X_i(0)\,\mathrm{e}^{(r-q_i+\omega_i)T+\mu_i x+\sigma_i\sqrt{x}\Phi^{-1}(p)}.$$

We find that K_i can be determined by using relations (26) and (27). ■

Because V_i is lognormal, the price $\mathrm{e}^{-rT}\mathbb{E}\left[\left(V_i - K_i\right)_+\right]$ can be expressed in a closed form using the Black & Scholes option pricing formula.

Theorem 4 *Consider a market where the assets follow the multivariate VG process (9). For $x > 0$, we have that*

$$\mathrm{e}^{-rT}\mathbb{E}\left[\left(S_x^c - K\right)_+\right] = \sum_{i=1}^{n} w_i \left(X_i(0)\,\mathrm{e}^{(\omega_i - q_i)T + \left(\mu_i + \frac{\sigma_i^2}{2}\right)x}\Phi\left(d_{i,1}\right) - K_i \mathrm{e}^{-rT}\Phi\left(d_{i,2}\right)\right), \tag{28}$$

where K_i is defined as in Theorem 3 and

$$d_{i,1} = \frac{\ln\left(\frac{X_i(0)}{K_i}\right) + \left(r - q_i + \omega_i\right)T + \mu_i x + \sigma_i^2 x}{\sigma_i\sqrt{x}}, \tag{29}$$

$$d_{i,2} = d_{i,1} - \sigma_i\sqrt{x}. \tag{30}$$

Proof. The random variable V_i has a lognormal distribution; see (24). This means that we can modify the Black & Scholes formula to price options on V_i with maturity T and strike K_i:

$$\mathrm{e}^{-rT}\mathbb{E}\left[(V_i - K_i)_+\right] = \mathrm{e}^{-rT}\left(X_i(0)\,\mathrm{e}^{(r-q_i+\omega_i)T+\mu_i x+\frac{\sigma_i^2 x}{2}}\Phi(d_{i,1}) - K_i\Phi(d_{i,2})\right) \tag{31}$$

where $d_{i,1}$ and $d_{i,2}$ are given by

$$\begin{aligned} d_{i,1} &= \frac{\ln\left(\frac{X_i(0)}{K_i}\right) + (r - q_i + \omega_i)T + \mu_i x + \sigma_i^2 x}{\sigma_i\sqrt{x}}, \\ d_{i,2} &= d_{i,1} - \sigma_i\sqrt{x}. \end{aligned}$$

We use Theorem 3 to decompose $\mathrm{e}^{-rT}\mathbb{E}\left[(S_x^c - K)_+\right]$ in terms of the marginal stop-loss premiums $\mathrm{e}^{-rT}\mathbb{E}\left[(V_i - K_i)_+\right]$, where the K_i follow from (26). We find that (28) holds by plugging (31) into expression (25). ■

4.2 A convex lower bound for the conditional sum

In this section, we search for a lower bound for $C[K;x]$. We replace the sum S_x by S_x^l, defined as follows

$$S_x^l = \mathbb{E}[S_x \mid \Lambda],$$

where

$$\Lambda = \sum_{j=1}^{n} \lambda_j \ln\frac{X_j}{X_j(0)} \mid G = x.$$

Note that the weights λ_j and thus also the r.v. Λ may depend on x. However, if no confusion is possible we will omit the dependence on x in the notation in order to keep the notation simple. Remark that other choices for Λ exist; see Deelstra et al. (2004). In Kaas et al. (2000), it is proven that the sum S_x^l is a convex lower bound for the sum S_x:

$$S_x^l \preceq_{cx} S_x. \tag{32}$$

As a result, we find from the definition in (1) that $C[K;x]$ can be bounded from below as follows:

$$\mathrm{e}^{-rT}\mathbb{E}\left[\left(S_x^l - K\right)_+\right] \leq C[K;x]. \tag{33}$$

In the following theorem, we formulate an explicit expression for the sum S_x^l.

Theorem 5 *Consider a market where the assets follow the multivariate VG process (9). For $x \geq 0$, the conditional basket S_x^l is given by the following expression:*

$$S_x^l \overset{\mathrm{d}}{=} \sum_{i=1}^{n} w_i X_i(0)\exp\left\{(r - q_i + \omega_i)T + \mu_i x + \frac{\sigma_i^2 x\left(1 - r_i^2\right)}{2} + r_i\sigma_i\sqrt{x}\Phi^{-1}(U)\right\}, \tag{34}$$

with $r_i = \operatorname{Corr}\left[\Lambda, \ln \frac{X_i}{X_i(0)} \mid G = x\right]$ *and*

$$\Lambda \stackrel{\mathrm{d}}{=} \sum_{j=1}^{n} \lambda_j \left((r - q_j + \omega_j) T + \mu_j x + \sigma_j \sqrt{x} B_j (1) \right). \tag{35}$$

The conditional variance $\operatorname{Var}\left[S_x^l\right]$ *is given by*

$$\operatorname{Var}\left[S_x^l\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j X_i (0) X_j (0) \mathrm{e}^{2rT + (\omega_i - q_i + \omega_j - q_j)T + \left(\mu_i + \mu_j + \frac{1}{2}\left(\sigma_i^2 + \sigma_j^2\right)\right)x} \left(\mathrm{e}^{r_i r_j \sigma_i \sigma_j x} - 1\right).$$

Proof. By using expression (12), we find that expression (35) holds for the random variable Λ. Furthermore, Λ has a normal distribution. Remark that for a bivariate normal distribution (X, Y) with $\rho = \operatorname{Corr}[X, Y]$, the r.v. $X \mid Y$ has again a normal distribution with mean:

$$\mathbb{E}[X \mid Y] = \mathbb{E}[X] + \rho \sqrt{\frac{\operatorname{Var}[X]}{\operatorname{Var}[Y]}} (Y - \mathbb{E}[Y]), \tag{36}$$

and variance $\operatorname{Var}[X] \left(1 - \rho^2\right)$. Using expression (36), we find that $\ln \frac{X_i}{X_i(0)} \mid G, \Lambda$ has a normal distribution with mean

$$\mathbb{E}\left[\ln \frac{X_i}{X_i(0)} \mid G = x, \Lambda\right] = (r - q_i + \omega_i) T + \mu_i x + r_i \sigma_i \sqrt{x} \left(\frac{\Lambda - \mathbb{E}[\Lambda]}{\sqrt{\operatorname{Var}[\Lambda]}} \right),$$

and variance

$$\operatorname{Var}\left[\ln \frac{X_i}{X_i(0)} \mid G = x, \Lambda\right] = \sigma_i^2 x \left(1 - r_i^2\right),$$

where $r_i = \operatorname{Corr}\left[\Lambda, \ln \frac{X_i}{X_i(0)} \mid G = x\right]$. We then find that

$$\mathbb{E}[X_i \mid G = x, \Lambda] = X_i (0) \exp\left\{ (r - q_i + \omega_i) T + \mu_i x + \frac{\sigma_i^2 x \left(1 - r_i^2\right)}{2} + r_i \sigma_i \sqrt{x} \Phi^{-1}(U) \right\},$$

which proves (34).

We can write the variance as follows

$$\begin{aligned} \operatorname{Var}\left[S_x^l\right] = & \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j X_i (0) X_j (0) \mathrm{e}^{2rT + (\omega_i - q_i + \omega_j - q_j)T + \left(\mu_i + \mu_j + \frac{1}{2}\left(\sigma_i^2\left(1 - r_i^2\right) + \sigma_j^2\left(1 - r_j^2\right)\right)\right)x} \\ & \times \operatorname{Cov}\left[\mathrm{e}^{r_i \sigma_i \sqrt{x} \Phi^{-1}(U)}, \mathrm{e}^{r_j \sigma_j \sqrt{x} \Phi^{-1}(U)}\right]. \end{aligned}$$

If we use that for any σ, $\mathbb{E}\left[\mathrm{e}^{\sigma \Phi^{-1}(U)}\right] = \mathrm{e}^{\frac{\sigma^2}{2}}$, we find that

$$\operatorname{Cov}\left[\mathrm{e}^{r_i \sigma_i \sqrt{x} \Phi^{-1}(U)}, \mathrm{e}^{r_j \sigma_j \sqrt{x} \Phi^{-1}(U)}\right] = \mathrm{e}^{\frac{1}{2}\left(r_i^2 \sigma_i^2 + r_j^2 \sigma_j^2\right)x} \left(\mathrm{e}^{r_i r_j \sigma_i \sigma_j x} - 1\right),$$

from which we find the desired result. ∎

In this paper, we choose the weights λ_j of the conditioning r.v. Λ by the 'maximal variance' method, proposed in Vanduffel et al. (2005). The convex order relation (32) implies that $\mathrm{Var}\left[S_x^l\right] \leq \mathrm{Var}[S_x]$. In Dhaene et al. (2002a), it was proven that $\mathrm{Var}\left[S_x^l\right] = \mathrm{Var}[S_x]$ implies that $S_x^l \overset{\mathrm{d}}{=} S_x$; see Cheung et al. (2013) for a generalization of this result. Using these observations, we choose λ_j such that $\mathrm{Var}\left[S_x^l\right]$ is as close as possible to $\mathrm{Var}[S_x]$. One can prove that the following choice is optimal:

$$\lambda_j = w_j X_j(0) \exp\left\{(r - q_j + \omega_j) T + \mu_j x + \frac{\sigma_j^2 x}{2}\right\}, \qquad \text{for } j = 1, 2, \ldots, n.$$

For other choices of the conditioning random variable Λ, we refer to Deelstra et al. (2004) and Deelstra & Hainaut (2014).

Remark 6 (Calculation of r_i) Take $x > 0$ and consider the conditional random variable S_x^l. If we denote the variance of Λ by $x\sigma_\Lambda^2$, we find from relations (7) and (35) that

$$\sigma_\Lambda^2 = \sum_{i=1}^{n} \lambda_i^2 \sigma_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \lambda_i \lambda_j \sigma_i \sigma_j \rho_{i,j}.$$

The correlation coefficient r_i is then given by

$$r_i = \frac{\sum_{j=1}^{n} \lambda_j \sigma_j \rho_{i,j}}{\sigma_\Lambda}.$$

A sufficient condition for r_i to be positive is that $\rho_{i,j} \geq 0$, for $i, j = 1, 2, \ldots, n$ and $i \neq j$.

∇

Define the random variable W_i as follows:

$$W_i = W_i(0) \exp\left\{(r - q_i + \omega_i) T + \mu_i x + \frac{\sigma_i^2 x \left(1 - r_i^2\right)}{2} + r_i \sigma_i \sqrt{x} \Phi^{-1}(U)\right\},$$

and $W_i(0) = X_i(0)$. Then we have that

$$\ln \frac{W_i}{W_i(0)} \overset{\mathrm{d}}{=} N\left((r - q_i + \omega_i) T + \mu_i x + \frac{\sigma_i^2 x \left(1 - r_i^2\right)}{2}, r_i^2 \sigma_i^2 x\right).$$

The sum S_x^l can be written as

$$S_x^l = \sum_{i=1}^{n} w_i W_i,$$

and given that (8) holds, S_x^l is a sum of n comonotonic lognormal random variables. Theorem 2 in Deelstra et al. (2004) proves that if assumption (8) does not hold, one can always find λ_i, $i = 1, 2, \ldots, n$ such that the corresponding r.v. S_x^l is a comonotonic sum.

Theorem 7 *Consider a market where the assets follow a multivariate VG process (9), where all pairwise correlations $\rho_{i,j}$ are non-negative. For $x \geq 0$, we have that*

$$e^{-rT}\mathbb{E}\left[\left(S_x^l - K\right)_+\right] = \sum_{i=1}^{n} w_i e^{-rT}\mathbb{E}\left[\left(W_i - K_i\right)_+\right], \tag{37}$$

where K_i is defined by

$$K_i = X_i(0)\exp\left\{(r - q_i + \omega_i)T + \mu_i x + \frac{\sigma_i^2 x\left(1 - r_i^2\right)}{2} + r_i\sigma_i\sqrt{x}\Phi^{-1}\left(F_{S_x^l}(K)\right)\right\}, \tag{38}$$

and $F_{S_x^l}(K)$ is determined such that the following relation holds:

$$\sum_{i=1}^{n} w_i K_i = K. \tag{39}$$

Proof. Because all pairwise correlations $\rho_{i,j}$ are non-negative, the sum S_x^l is a weighted sum of the comonotonic random variables $W_1, W_2, \ldots, W_n$. Each W_i has a lognormal distribution and its inverse cdf $F_{W_i}^{-1}(p)$ is given by

$$F_{W_i}^{-1}(p) = X_i(0)\exp\left\{(r - q_i + \omega_i)T + \mu_i x + \frac{\sigma_i^2 x\left(1 - r_i^2\right)}{2} + r_i\sigma_i\sqrt{x}\Phi^{-1}(p)\right\}.$$

The marginals F_{W_i} are strictly increasing and continuous, from which we find that also $F_{S_x^l}$ is strictly increasing and continuous. If we apply Theorem 1, we find that (37) holds and that the K_i can be determined using the relations (38) and (39). ■

Because the right hand side of expression (37) consists of stop-loss premiums of lognormal random variables, we can write them in an analytical way by applying the Black & Scholes option pricing formula.

Theorem 8 *Consider a market where the assets follow the multivariate VG process (9), where all pairwise correlations $\rho_{i,j}$ are non-negative. For $x \geq 0$, we have that*

$$\mathrm{e}^{-rT}\mathbb{E}\left[\left(S_x^l - K\right)_+\right] = \sum_{i=1}^{n} w_i\left(X_i(0)\,\mathrm{e}^{(\omega_i - q_i)T + \left(\mu_i + \frac{\sigma_i^2}{2}\right)x}\Phi\left(d_{i,1}\right) - K_i\mathrm{e}^{-rT}\Phi\left(d_{i,2}\right)\right), \tag{40}$$

where K_i is defined as in Theorem 7 and

$$d_{i,1} = \frac{\ln\frac{X_i(0)}{K_i} + (r - q_i + \omega_i)T + \mu_i x + \frac{\sigma_i^2 x\left(1 + r_i^2\right)}{2}}{r_i\sigma_i\sqrt{x}}, \tag{41}$$

$$d_{i,2} = d_{i,1} - \sigma_i r_i\sqrt{x}. \tag{42}$$

Proof. The sum S_x^l is the sum of the comonotonic r.v.'s $W_1, W_2, \ldots, W_n$ and each W_i has a lognormal distribution. The stop-loss premium $\mathbb{E}\left[\left(W_i - K_i\right)_+\right]$ is given by

$$\mathrm{e}^{-rT}\mathbb{E}\left[\left(W_i - K_i\right)_+\right] = \mathrm{e}^{-rT}\left(W_i(0)\,\mathrm{e}^{(r - q_i + \omega_i)T + \mu_i x + \frac{\sigma_i^2 x\left(1 - r_i^2\right)}{2} + \frac{\sigma_i^2 r_i^2 x}{2}}\Phi\left(d_{i,1}\right) - K_i\Phi\left(d_{i,2}\right)\right),$$

from which we find that (40) holds, where $d_{i,1}$ and $d_{i,2}$ are given by (41) and (42), respectively. ■

5 An approximate basket option price

In this section, we first derive the approximation $\overline{C}[K]$ for the basket option price $C[K]$. Afterwards, the unconditional basket option price is found by approximating the integral by a Gauss-Laguerre quadrature formula.

5.1 A moment-matching approximation for the basket option price

In this subsection, we derive the approximate basket option price $\overline{C}[K]$ for $K \geq 0$ which is based on the conditional upper and lower bounds derived in Section 4. First, the conditional approximation $\overline{C}[K;x]$ is defined as

$$\overline{C}[K;x] = \mathrm{e}^{-rT}\left(z_x \mathbb{E}\left[\left(S_x^l - K\right)_+\right] + (1-z_x)\,\mathbb{E}\left[(S_x^c - K)_+\right]\right), \tag{43}$$

where $z_x \in [0,1]$. Then, $\overline{C}[K;x]$ can be interpreted as the price of a call option with strike K, written on a synthetic conditional stock market basket, denoted by $\overline{S}_x$, in the sense that the following relation holds

$$\overline{C}[K;x] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(\overline{S}_x - K\right)_+\right], \text{ for all } K \geq 0.$$

The weights z_x are chosen such that the approximation $\overline{S}_x$ is as close as possible to S_x, defined in (14). We follow the approach of Linders (2013) and take z_x as follows

$$z_x = \frac{\mathrm{Var}\left[S_x^c\right] - \mathrm{Var}\left[S_x\right]}{\mathrm{Var}\left[S_x^c\right] - \mathrm{Var}\left[S_x^l\right]}, \tag{44}$$

where $\overline{C}[K;x]$ is given by (43) and where z_x is chosen as in (44). Remark that the synthetic random variable $\overline{S}_x$ always exists and is unique because of the one-to-one relationship between a convex call option curve and its stop-loss premium.

Second, we define the approximate unconditional basket option price $\overline{C}[K]$ as follows

$$\overline{C}[K] = \int_0^{+\infty} \overline{C}[K;x]\, f_G(x)\,\mathrm{d}x, \tag{45}$$

We can then state the following theorem. For a proof of this theorem, we refer to the appendix.

Theorem 9 *Consider a market where the assets follow the multivariate VG model (9) and let $\overline{C}[K]$ be defined by (45). Then there exists a r.v. $\overline{S}$ such that*

$$\overline{C}[K] = e^{-rT}\mathbb{E}\left[\left(\overline{S} - K\right)_+\right],$$

and moreover,

$$Var\left[\overline{S}\right] = Var[S].$$

Note that upper and lower bounds for the basket option price $C[K]$ can be obtained by choosing $z_x = 0$ or $z_x = 1$, respectively.

5.2 Numerical integration

The expression (45) for the approximate basket option price is given in an integral form and has to be evaluated using numerical integration procedures. In this subsection, we show that by using the Gauss-Laguerre quadrature, we arrive at a simple and easy-to-implement approximation for $\overline{C}[K]$.

The price $C[K]$ for a basket option with maturity T and strike K is approximated by $\overline{C}[K]$, which can be written as in (45) where f_G is given by expression (6). Plugging this expression for f_G into (45) results in

$$\overline{C}[K] = \int_0^{+\infty} \overline{C}[K;x] \frac{\left(\frac{1}{\nu}\right)^{\frac{1}{\nu}T}}{\Gamma\left(\frac{1}{\nu}T\right)} x^{\frac{1}{\nu}T-1} \exp\left\{-x\frac{1}{\nu}\right\} \mathrm{d}x.$$

Using the substitution $y = \frac{x}{\nu}$ and defining $\beta = \frac{1}{\Gamma\left(\frac{1}{\nu}T\right)}$ and $\alpha = \frac{1}{\nu}T - 1$, we can rewrite the approximation $\overline{C}[K]$ as follows:

$$\overline{C}[K] = \beta \int_0^{+\infty} \overline{C}[K;\nu y]\, g(y)\, \mathrm{d}y, \tag{46}$$

with

$$g(y) = y^{\alpha} \mathrm{e}^{-y}. \tag{47}$$

Expression (46) shows that the approximation $\overline{C}[K]$ can be expressed as an integral, where the integrand is the product of the weighting function g and the smooth function $\overline{C}$. Because the weight function is given by (47) with $\alpha > -1$, we can use the generalized Gauss-Laguerre polynomial of degree d, denoted by $L_d^{(\alpha)}$, to approximate the integral in expression (46). This approach results in the following approximation for $\overline{C}[K]$:

$$\overline{C}[K] \approx \beta \sum_{i=1}^{d} g_{i:d} \overline{C}[K;\nu x_{i:d}],$$

where $g_{i:d}$ are the quadrature weights and $x_{i:d}$ is the i^{th} root of the Gauss-Laguerre polynomial $L_d^{(\alpha)}$. A similar approach for determining joint defaults and pricing CDO tranches was followed in Garcia & Goossens (2010).

The function values and the derivative of the Gauss-Laguerre polynomial $L_d^{(\alpha)}$ are given by the following recurrence relations:

$$\begin{aligned} (d+1)\, L_{d+1}^{(\alpha)}(y) &= (2d+1+\alpha-y)\, L_d^{(\alpha)}(y) - (d-\alpha)\, L_{d-1}^{(\alpha)}(y), \\ x\frac{\mathrm{d}L_d^{(\alpha)}}{\mathrm{d}x}(x) &= dL_d^{(\alpha)}(x) - (d+\alpha)\, L_{d-1}^{(\alpha)}(x), \end{aligned}$$

with starting values $L_0^{(\alpha)}(y) = L_{-1}^{(\alpha)}(y) = 0$. The roots $x_{i:d}$ can be determined using a Newton-Raphson iteration. The starting value $x_{1:d}^{(1)}$ to determine the first root $x_{1:d}$ is given by

$$x_{1:d}^{(1)} = \frac{(1+\alpha)(3+0.92\alpha)}{1+2.4n+1.8},$$

and for the second root $x_{2:d}$ the starting value is

$$x_{2:d}^{(1)} = x_{1:d} + \frac{15+6.25\alpha}{1+0.9\alpha+2.5d}.$$

For the other roots $x_{i:d}$, $i = 2, 3, \ldots, d$, we use the starting value $x_{i:d}^{(1)}$:

$$x_{i:d}^{(1)} = x_{i-1:d} + \left(\frac{1+2.55(i-2)}{1.9(i-2)} + \frac{1.26(i-2)\alpha}{1+3.5(i-2)} \right) \frac{x_{i-1:d} - x_{i-2:d}}{1+0.3\alpha}.$$

Using the starting value $x_{i:d}^{(1)}$, the root $x_{i:d}$ of the function $L_d^{(\alpha)}$ is determined by using the following iterations

$$x_{i:d}^{(k+1)} = x_{i:d}^{(k)} - \frac{L_d^{(\alpha)}\left(x_{i:d}^{(k)}\right)}{\frac{\mathrm{d}L_d^{(\alpha)}}{\mathrm{d}x}\left(x_{i:d}^{(k)}\right)}.$$

The quadrature weights $g_{i:d}$ are given by:

$$g_{i:d} = \frac{-\Gamma(\alpha+d)}{d\Gamma(d)\, L_{d-1}^{(\alpha)}(x_{i:d}) \frac{\mathrm{d}L_d^{(\alpha)}}{\mathrm{d}x}(x_{i:d})}.$$

Note that given $\Gamma(\alpha+1)$, we can evaluate $\Gamma(\alpha+d)$ using the relation $\Gamma(x) = (x-1)\Gamma(x-1)$, which implies that the Gamma function has to be evaluated only once. Alternatively, one can determine the weights and the roots of $L_d^{(\alpha)}$ using an eigenvalue problem; see e.g. Press et al. (1992).

6 Numerical illustration

The bounds derived in this paper can be used to approximate the value of a basket option when the stock price dynamics are described by the multivariate VG process (9). We consider a basket option written on three underlying non-dividend paying stocks where r is chosen to be 3%. Furthermore, we assume that conditional on the time change, the stock prices are independent, i.e. $\rho_{i,j} = 0$ for $i \neq j$ and $i, j = 1, 2, 3$. The model parameters and the weights are given in Table 1. For a number of strikes K, we compare the approximate option price $\overline{C}[K]$ with its corresponding Monte Carlo estimate, denoted by $C^{sim}[K]$. Monte Carlo estimates are determined using 10^6 simulated values. The approximations $\overline{C}[K]$ are determined by using Gauss-Laguerre polynomials with degree $d = 24$. The quality of the approximation is measured by the relative error $\varepsilon[K]$:

$$\varepsilon[K] = \frac{\left|\overline{C}[K] - C^{sim}[K]\right|}{C^{sim}[K]}.$$

We determine option prices for the maturities $T = 2/12$, 1 and 2 and $\nu = 0.5$ and 0.9. The results are displayed in Table 2. The approximate basket option curve can directly be linked to the probability distribution (pdf) $f_{\overline{S}}$ of the approximate random variable $\overline{S}$. Indeed, we have that

$$f_{\overline{S}}(K) = \mathrm{e}^{rT} \frac{\partial \overline{C}^2 [K]}{\partial K^2}, \tag{48}$$

provided the second derivative of the basket option curve $\overline{C}$ exists. In case we replace the approximate basket option curve $\overline{C}$ in (48) with the corresponding Monte Carlo curve C^{sim}, we find the empirical pdf $f_{S^{sim}}$ of the simulated random variable S^{sim}. A plot of $f_{\overline{S}}$ and $f_{S^{sim}}$ is shown in Figure 1, where we have taken $\nu = 0.5$ and $T = 2$. A similar plot but now for $\nu = 0.9$ and $T = 1$ is shown in Figure 2. We conclude that a higher value of ν leads to marginal distributions that are more skewed.

Consider the three-stock basket option with parameters given in Table 1 and $\nu = 0.5$. In Table 3, we compare the approximation $\overline{C}[K]$ with the Monte Carlo estimate $C^{sim}[K]$, for $\sigma_1 = 0.05$, 0.25 and 0.75. We observe that the error is larger when σ_1 is large. A comparison between the probability distribution functions $f_{\overline{S}}$ and $f_{S^{sim}}$ for $\sigma_1 = 0.75$ and $T = 2$ is given in Figure 3. In order to investigate the effect of σ_1 on the quality of the approximation, we also determine $\overline{C}[300]$ and $C^{sim}[300]$ for an at-the-money basket option where σ_1 varies between 0 and 1. Figure 4 depicts the relative error in function of σ_1. In case σ_1 is larger than 0.5, the relative error is rising above 1%, whereas the approximation $\overline{C}[300]$ proves to be accurate when σ_1 is below 0.5. For all values of σ_1, the relative error stays below 5% and we conclude that using the approximation $\overline{C}[K]$ for $C[K]$ is justified in these situations.

Table 4 reports $\overline{C}[K]$ and the corresponding Monte Carlo estimates $C^{sim}[K]$ for the situation where $\nu = 0.5$ and $\mu_1 =$ -1/5, -1 and -0.5. The other parameters remain unchanged and are listed in Table 1. From these results, we conclude that for varying values of μ_1, the approximation $\overline{C}[K]$ always remains accurate. In order to further assess the impact of the parameter μ_1, we set $T = 1$ and $\nu = 0.5$ and determine the approximations $\overline{C}[300]$ and $C^{sim}[300]$ for an at-the-money basket option for different choices of μ_1. A graph of the relative error in function of μ_1 is shown in Figure 5.

Table 1: Input for the 3-stock basket option

	1	2	3
μ_i	-0.15	-0.06	-0.2
σ_i	0.1	0.2	0.04
$X_i(0)$	100	100	100
w_i	1	1	1

Table 2: Call option prices for the three-stock basket option outlined in Table 1.

T	ν	K	$\overline{C}[K]$	$C^{sim}[K]$	$\varepsilon[K]$
2 months	0.5	225	77.6590	77.6565	0.003%
		270	33.4817	33.4764	0.02%
		300	6.7475	6.7396	0.12%
		330	0.0186	0.0189	1.59%
	0.9	225	77.7958	77.8026	0.01%
		270	33.9759	33.9793	0.01%
		300	7.1060	7.0938	0.17%
		330	0.0168	0.0170	1.18%
1 year	0.5	225	91.0976	91.1047	0.01%
		270	49.5413	49.5464	0.01%
		300	25.4644	25.4671	0.01%
		330	8.1233	8.1273	0.05%
		375	0.1804	0.1844	2.17%
	0.9	225	91.7094	91.7206	0.01%
		270	51.2344	51.2350	0.001%
		300	27.6608	27.6558	0.02%
		330	9.5987	9.5991	0.004%
		375	0.1429	0.1463	2.32%
2 years	0.5	225	107.2349	107.2281	0.0063%
		270	67.4772	67.4667	0.02%
		300	43.9728	43.9594	0.03%
		330	24.7395	24.7249	0.06%
		375	6.7266	6.7322	0.08%
	0.9	225	108.2324	108.2263	0.01%
		270	69.8255	69.8113	0.02%
		300	47.1523	47.1334	0.04%
		330	28.1138	28.0975	0.06%
		375	8.6410	8.6369	0.05%

Table 3: Call option prices for the three-stock basket option outlined in Table 1 for $\nu = 0.5$ and different choices of σ_1.

T	σ_1	K	$\overline{C}[K]$	$C^{sim}[K]$	$\varepsilon[K]$
1 year	0.05	225	91.0626	91.0778	0.02%
		270	49.3877	49.3984	0.02%
		300	25.1855	25.1867	0.00%
		330	7.7972	7.7982	0.01%
		375	0.1352	0.1379	1.96%
	0.25	225	91.3247	91.3072	0.02%
		270	50.5040	50.4879	0.03%
		300	27.2564	27.2453	0.04%
		330	10.4273	10.4261	0.01%
		375	0.9801	0.9816	0.15%
	0.75	225	92.9322	92.9341	0.00%
		270	57.1651	56.8966	0.47%
		300	39.3118	38.8472	1.20%
		330	27.0538	26.5524	1.89%
		375	16.8200	16.4541	2.22%
2 years	0.05	225	107.1665	107.2119	0.04%
		270	67.2492	67.2862	0.05%
		300	43.5964	43.6230	0.06%
		330	24.2520	24.2661	0.06%
		375	6.3097	6.3152	0.09%
	0.25	225	107.6690	107.6594	0.01%
		270	68.8704	68.8677	0.00%
		300	46.3130	46.3229	0.02%
		330	27.9332	27.9518	0.07%
		375	10.0106	10.0240	0.13%
	0.75	225	110.6198	110.3766	0.22%
		270	78.1630	77.3573	1.04%
		300	61.7599	60.4695	2.13%
		330	49.3989	47.8211	3.30%
		375	36.7328	35.1726	4.44%

Table 4: Call option prices for the three-stock basket option outlined in Table 1 for $\nu = 0.5$ and different choices of μ_1.

T	μ_1	K	$\overline{C}[K]$	$C^{sim}[K]$	$\varepsilon[K]$
1 year	-1.5	225	96.3776	96.4354	0.06%
		270	63.8382	63.8155	0.04%
		300	46.2160	46.3943	0.38%
		330	32.6219	32.3818	0.74%
		375	16.7600	17.1121	2.06%
	-1	225	94.4482	94.4175	0.03%
		270	59.0404	58.9765	0.11%
		300	39.6293	39.6573	0.07%
		330	24.4614	24.3639	0.40%
		375	9.0462	9.0840	0.42%
	-0.05	225	90.9049	90.9032	0.00%
		270	48.5939	48.5907	0.01%
		300	23.7434	23.7351	0.03%
		330	6.4399	6.4335	0.10%
		375	0.1708	0.1745	2.12%
2 years	-1.5	225	114.9621	114.9355	0.02%
		270	85.5939	85.6799	0.10%
		300	69.9588	70.0723	0.16%
		330	57.4576	57.2037	0.44%
		375	42.0287	42.0974	0.16%
	-1	225	112.4230	112.4395	0.01%
		270	80.0549	80.0030	0.06%
		300	62.0574	62.1170	0.10%
		330	47.2844	47.2695	0.03%
		375	30.2664	30.2235	0.14%
	-0.05	225	106.8760	106.9065	0.03%
		270	66.1511	66.1746	0.04%
		300	41.7363	41.7491	0.03%
		330	21.9395	21.9398	0.00%
		375	4.8395	4.8463	0.14%

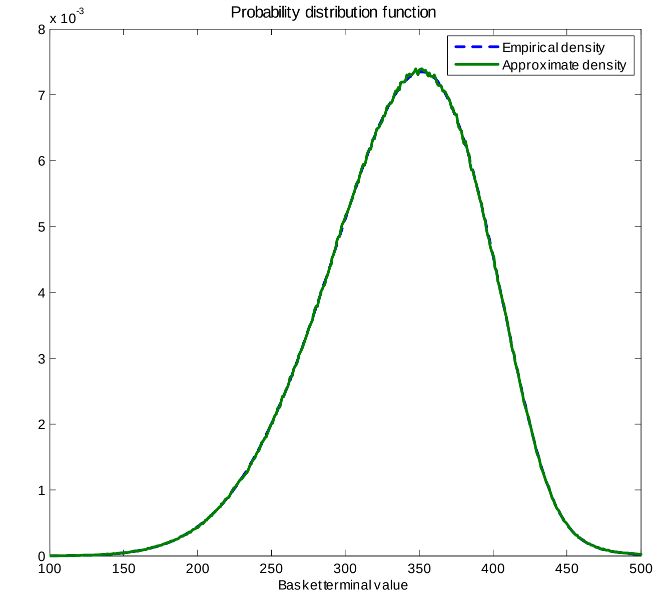

Figure 1: The probability distribution function for the simulated sum S^{sim} (solid line) and the approximate sum $\overline{S}$ (dashed line) with $\nu = 0.5$ and $T = 2$.

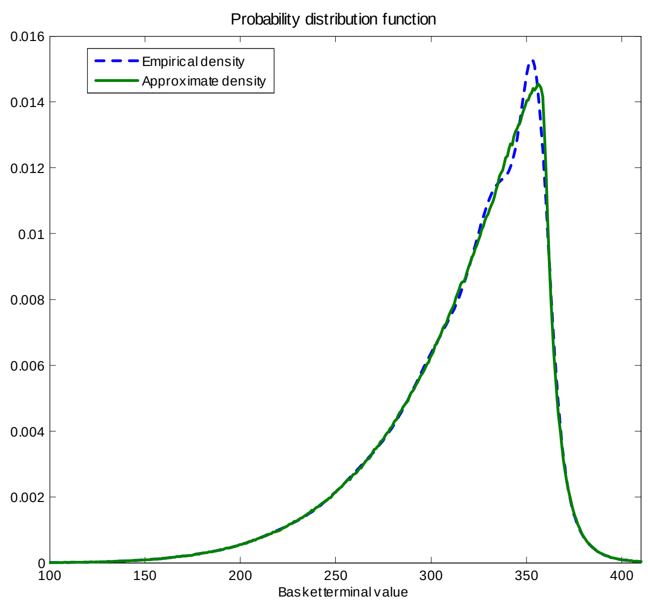

Figure 2: The probability distribution function for the simulated sum S^{sim} (solid line) and the approximate sum $\overline{S}$ (dashed line) with $\nu = 0.9$ and $T = 1$.

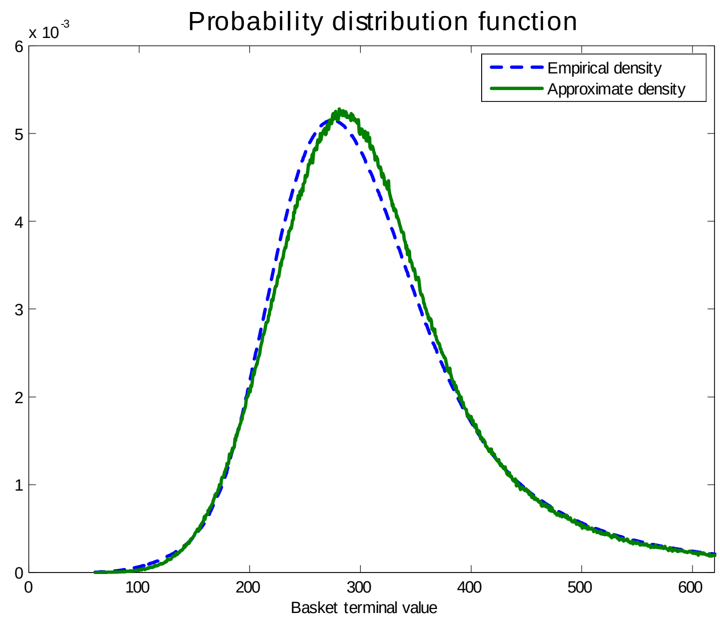

Figure 3: The probability distribution function for the simulated sum S^{sim} (solid line) and the approximate sum $\overline{S}$ (dashed line) with $\sigma_1 = 0.75$ and $T = 2$.

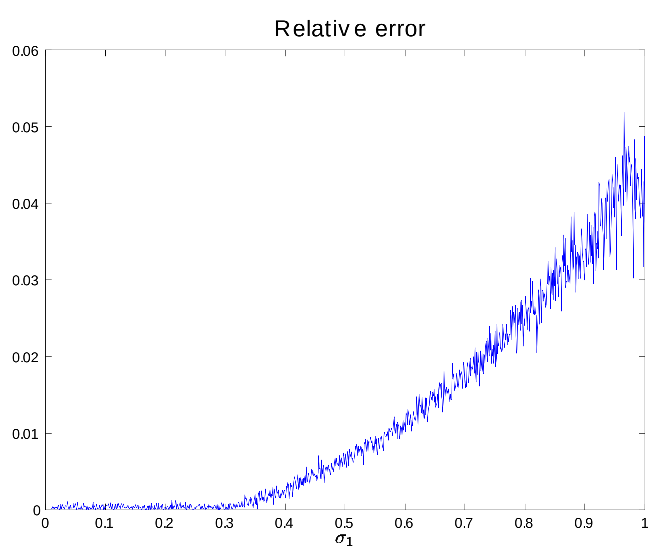

Figure 4: Relative error for different choices of σ_1.

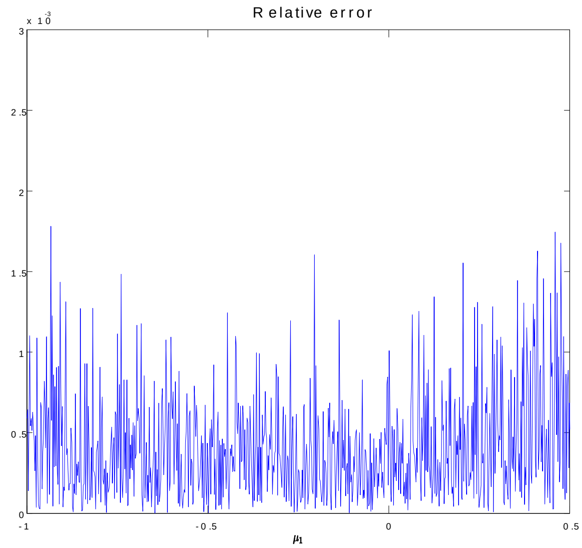

Figure 5: Relative error for different choices of μ_1.

7 Calibration and pricing of basket options

The perfect (multivariate) stock price model does not exist and any model has its merits, but also its drawbacks. Therefore, the goal is to find a simple and intuitive model which is also capable of giving a correct description of the stock price behavior. In the multivariate VG model (9), the individual stocks are modeled using a Variance Gamma model and it is well-known that this model provides a good fit of the observed vanilla option curves. In order to test if the multivariate VG model is a good model for pricing multi-asset derivatives, we investigate in this section if the model can closely match quoted basket option prices. We illustrate this methodology using the Dow Jones Industrial Average (DJ). This price weighted stock market index consists of 30 stocks and on each component of the DJ and on the DJ itself, options are traded and their prices are observable. The price of the vanilla option written on stock i with strike K and maturity T is denoted by $C_i\left[K,T\right]$. Note, however, that vanilla options are of American type, whereas DJ index options are of European type.

We denote by X_i, $i=1,2,\ldots,30$, the price level of the $i-$th component of the DJ at time T. If the price vector $(X_1,X_2,\ldots,X_{30})$ is described by the multivariate VG model (9), the log returns of stock i can be described by a Variance Gamma distribution with parameters (σ_i,ν,μ_i). In Carr & Madan (1999), the authors show how to determine the Variance Gamma price $C_i^{VG}\left[K,T\right]$ for an option on stock i by using the Fast Fourier Transform (FFT). For a series of traded strikes K, we can observe the call option prices $C_i\left[K,T\right]$ and the parameters $\mu_1,\mu_2,\ldots,\mu_{30},\sigma_1,\sigma_2,\ldots,\sigma_{30},\nu$ can be determined by simultaneously calibrating the vanilla option curves. For April 18, 2008, the calibrated parameters together with the calibration error are listed in Table 5, where we used the available vanilla option prices with a time to maturity of 64 days.

Given the vanilla option curves, all parameters of the multivariate VG model (9) can be calibrated, except the correlation parameters $\rho_{i,j}$. Here, we make the simplifying assumption that all pairwise correlations are equal, i.e.

$$\rho_{i,j} = \rho, \text{ if } i \neq j,$$

for some $\rho \geq 0$. Index options on the DJ are traded and their observed prices are denoted by $C[K, T]$ for strike K and maturity T. The model price of an index option under the multivariate VG model (9) is denoted by $C^{VG}\left[K, T; \underline{\mu}, \underline{\sigma}, \nu, \rho\right]$. Note that this price can be closely approximated using the techniques described in Section 5. This approximate DJ index option price is denoted by $\overline{C}^{VG}\left[K, T; \underline{\mu}, \underline{\sigma}, \nu, \rho\right]$. The parameters $\underline{\mu}$, $\underline{\sigma}$ and ν are calibrated to the vanilla option curves in a first step of the calibration. However, for any choice of ρ, another index price arises.

The parameter ρ can be calibrated if multi-asset derivatives are traded. The availability of market quotes for index options together with an approximate basket option pricing formula enable us to determine an implied estimate for ρ by minimizing the relative error between the market and model index option quotes. Moreover, this second step of the calibration procedure is relatively fast because for a given set of parameters, basket options can be priced in a fast way. The implied average correlation for April 18, 2008 is listed in Table 6. Using the calibrated marginal parameters shown in Table 5 and the calibrated joint parameters ν and ρ listed in Table 6, we can determine the DJ index option price $\overline{C}^{VG}\left[K, T; \underline{\mu}, \underline{\sigma}, \nu, \rho\right]$ and compare this price with the market price $C[K, T]$ for any traded strike K; see Figure 6. We find that the multivariate VG model is capable of closely matching the observed market quotes for DJ options. The relative error and the RMSE are shown in Table 5.

The implied average correlation is also computed for 2 other dates, namely May 22 and July 18, 2008, respectively. These values are depicted in Table 6. In Figure 7, we compare again the DJ index option price with the market price for a series of strikes K. Both graphs show that by choosing equal pairwise correlations, the multivariate Variance Gamma model is able to fit the observed market option prices remarkably well.

8 Conclusion

The rapid growth of financial markets over the past decades has increased the interest in multi-asset products, such as basket options. In order to price such products, one needs to model the joint dynamics of a number of dependent stock prices. In this paper, we proposed a methodology for pricing basket options where the stock prices composing the basket are modeled by a multivariate Variance Gamma model. The individual stock prices are then modeled using a Variance Gamma process, but they are dependent through a common time change. Conditional on the time change, the basket is a weighted sum of correlated geometric Brownian motions. Using the additivity property of comonotonic stop-loss premiums, we derived a closed-form expression for the approximate basket option price as a linear combination of Black & Scholes prices, where the weights and values of the subordinator were based on Gauss-Laguerre polynomials.

Table 5: Calibrated marginal parameters for the multivariate VG model on April 18, 2008.

	$X_i(0)$	σ_i	μ_i	RMSE
Alcoa Incorporated	36.26	0.5374	-0.50720	0.61%
American Express Company	45.53	0.3715	-1.18450	1.99%
American International group	48.23	0.4076	-1.85920	4.69%
Bank of America	38.56	0.4256	-1.30810	2.63%
Boeing Corporation	78.66	0.3640	-0.68050	3.70%
Caterpillar	85.28	0.3731	-0.71440	2.81%
JP Morgan	45.76	0.3490	-0.64090	4.65%
Chevron	93.18	0.2168	-0.48380	1.53%
Citigroup	25.11	0.4227	-0.65850	6.17%
Coca Cola Company	60.11	0.2710	-0.52720	6.55%
Walt Disney Company	31.33	0.2962	-0.55880	2.26%
DuPont	52.02	0.3222	-0.50080	1.14%
Exxon Mobile	94.00	0.2646	-0.59700	8.08%
General Electric	32.69	0.2327	-0.28010	4.18%
General Motors	20.13	0.6881	-1.33890	4.01%
Hewlet-Packard	48.18	0.3927	-0.62160	1.01%
Home Depot	28.68	0.4451	-1.08610	0.50%
Intel	22.55	0.3652	-0.76170	1.82%
IBM	124.40	0.2461	-0.60900	6.43%
Johnson & Johnson	66.51	0.1775	-0.29690	2.86%
McDonald's	58.30	0.2122	-0.43760	1.79%
Merck & Company	39.76	0.4160	-0.91710	8.62%
Microsoft	30.00	0.3407	-0.67170	1.59%
3M	82.90	0.2608	-0.45860	1.12%
Pfizer	20.47	0.2156	0.33030	2.84%
Practer & Gamble	67.17	0.1916	-0.44340	1.55%
AT&T	37.51	0.3172	-0.71230	0.72%
United Technologies	72.51	0.3082	-0.68880	3.06%
Verizon	36.03	0.3141	-0.65150	1.04%
Wal-Mart Stores	56.31	0.2112	-0.37380	1.17%

Table 6: Calibrated parameters for the DJ on 3 different days. The RMSE and the relative error are determined by comparing the observed DJ index option curve with the corresponding model prices; see also Figure 6 and 7.

day	April 18, 2008	May 22, 2008	July 18, 2008
Time to maturity	64 days	30 days	29 days
ν	0.076312	0.04380	0.03395
ρ	0.064745	0.23293	0.21057
RMSE	0.0796	0.1311	0.0367
Relative error	0.0154	0.0574	0.0401

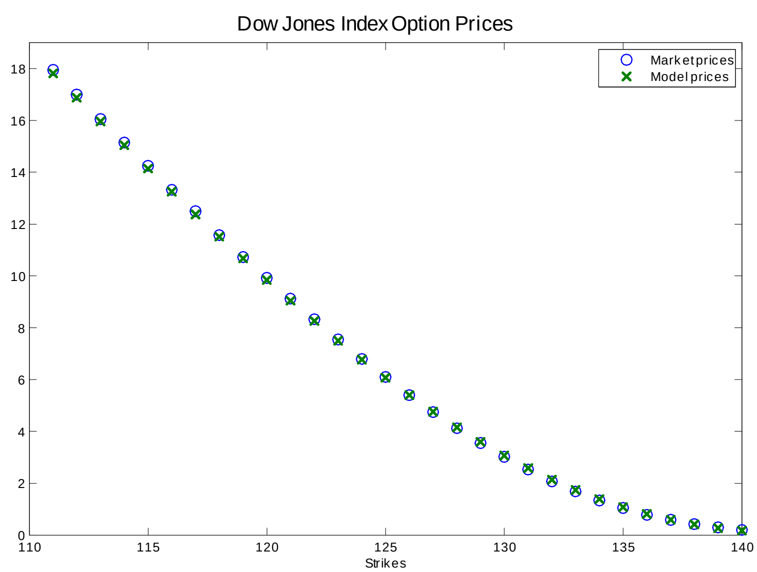

Figure 6: Model (crosses) and market (circles) prices for traded Dow Jones Index options on April 18, 2008 and time to maturity equal to 64 days.

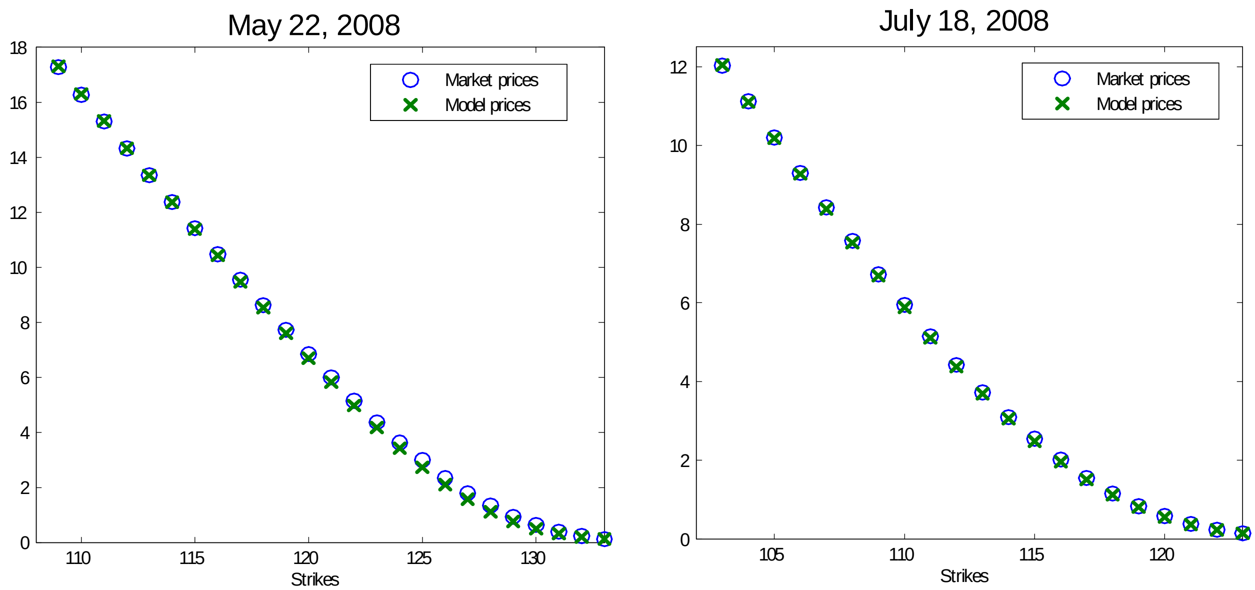

Figure 7: Model (crosses) and market (circles) prices for traded Dow Jones Index options on July 18, 2008 and May 22, 2008. The time to maturity is equal to 29 and 30 days, respectively.

In this paper, we searched for a pricing formula that gives us a reasonable compromise between the complexity and the tractability of the underlying model. Complexity here is understood as the ability of the underlying model to capture the characteristics of single-asset and multi-asset option prices, whereas tractability is related to the calibration of the model. Once a pricing formula is available and single-asset as well as multi-asset derivatives are traded, the model can be calibrated. Therefore, the availability of market quotes for options on the index and its components together with an approximate basket option pricing formula enabled us to have an efficient and fast calibration of the multivariate Variance Gamma model. Two steps were carried out in the calibration procedure. Firstly, the marginal parameters and the distribution of the common time change were calibrated to market quotes of the vanilla options by using the Carr-Madan formula. Secondly, we determined an implied estimate for the average correlation by minimizing the relative error between the market and model index option quotes.

As an illustration, we showed that the multivariate VG model is able to closely match the observed Dow Jones index options. By assuming an equal pairwise correlation between the different stocks, we observed that the model still provides us with an accurate fit as well as a good estimate for the implied correlation.

9 Appendix

Proof of Theorem 9. We first prove that the optimal value for z_x is given by (44). Choosing the approximation $\overline{S}_x$ as close as possible to S_x, leads to the following criterion,

$$\int_{-\infty}^{+\infty} \left(\mathbb{E}\left[\left(\overline{S}_x - K\right)_+\right] - \mathbb{E}\left[(S_x - K)_+\right]\right) \mathrm{d}K = 0$$

$$\iff \int_{-\infty}^{+\infty} \left(z_x \mathbb{E}\left[\left(S_x^l - K\right)_+\right] + (1 - z_x)\, \mathbb{E}\left[(S_x^c - K)_+\right] - \mathbb{E}\left[(S_x - K)_+\right]\right) \mathrm{d}K = 0$$

$$\begin{aligned}\iff z_x &\left[2\int_{-\infty}^{+\infty} \left(\mathbb{E}\left[(S_x^c - K)_+\right] - \left(\mathbb{E}\left[S_x^c\right] - K\right)_+\right) \mathrm{d}K - 2\int_{-\infty}^{+\infty} \left(\mathbb{E}\left[\left(S_x^l - K\right)_+\right] - \left(\mathbb{E}\left[S_x^l\right] - K\right)_+\right) \mathrm{d}K\right] \\ &= \int_{-\infty}^{+\infty} \left(\mathbb{E}\left[(S_x^c - K)_+\right] - \left(\mathbb{E}\left[S_x^c\right] - K\right)_+\right) \mathrm{d}K - \int_{-\infty}^{+\infty} \left(\mathbb{E}\left[(S_x - K)_+\right] - \left(\mathbb{E}\left[S_x\right] - K\right)_+\right) \mathrm{d}K\end{aligned}$$

$$\iff z_x = \frac{\mathrm{Var}\left[S_x^c\right] - \mathrm{Var}\left[S_x\right]}{\mathrm{Var}\left[S_x^c\right] - \mathrm{Var}\left[S_x^l\right]},$$

where we use $\mathbb{E}\left[S_x^c\right] = \mathbb{E}\left[S_x\right] = \mathbb{E}\left[S_x^l\right]$ and the following variance relation,

$$\mathrm{Var}\left[X\right] = 2\int_{-\infty}^{+\infty} \left(\mathbb{E}\left[(X - K)_+\right] - \left(\mathbb{E}\left[X\right] - K\right)_+\right) \mathrm{d}K.$$

We then have

$$\begin{aligned}\mathrm{Var}\left[\overline{S}\right] - \mathrm{Var}\left[S\right] &= \int_0^{+\infty} \left(\mathrm{Var}\left[\overline{S}_x\right] - \mathrm{Var}\left[S_x\right]\right) f_G\left(x\right) \mathrm{d}x \\ &= \int_0^{+\infty} \left[2\int_{-\infty}^{+\infty} \left(\mathbb{E}\left[\left(\overline{S}_x - K\right)_+\right] - \mathbb{E}\left[(S_x - K)_+\right]\right) \mathrm{d}K\right] f_G\left(x\right) \mathrm{d}x \\ &= 0.\end{aligned}$$

■

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